Fall 2023 Qualifying Exam
Decision-Sciences and Operations Research

GENERAL INSTRUCTIONS:

1. Answer each question in a separate book.
2. Indicate on the cover of *each* book your code number. *Do not write your name on any answer book.* On *one* of your books list the numbers of *all* the questions answered.
3. Return all answer books in the folder provided. Additional answer books are available if needed.

SPECIFIC INSTRUCTIONS:

Answer any 5 out of the 6 following questions to the best of your ability.

POLICY ON MISPRINTS AND AMBIGUITIES:

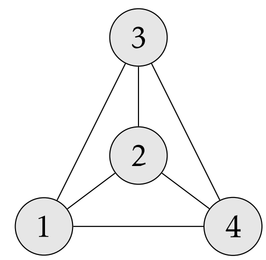
The Exam Committee tries to proofread the exam as carefully as possible. Nevertheless, the exam sometimes contains misprints and ambiguities. If you are convinced a problem has been stated incorrectly, mention this to the proctor. If necessary, the proctor can contact a representative of the area to resolve problems. In any case, you should indicate your interpretation of the problem in your written answer. Your interpretation should be such that the problem is nontrivial.

1. **Optimization Modeling**

   Two-Out-Of-Three in Triangles. This sounds like a made up problem, but it arose as part of the question-writer's research. We are given an undirected graph $G = (V, E)$. The graph contains a collection of triangles $\mathcal{T} = \{T_1, T_2, \dots T_q\}$ (cliques of size 3). We would like to select a *maximum* subset of the graph's triangles so that the nodes can be partitioned into two sets $(S, V \setminus S)$ with the property that each selected triangle has *exactly* two nodes in S.

   For example, consider the clique of size 4. It has four triangles:

   $$\mathcal{T} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}.$$

   If we select triangles $\{1,2,3\}$ and $\{1,3,4\}$, then by setting $S = \{1,3\}$, each of the two selected triangles has exactly two nodes in S. However, it is impossible to select three of these triangles and partition the nodes to satisfy the required property. Thus, the optimal solution to this instance has cardinality 2.

   (a) Write an integer programming formulation for the general version of this problem. Be sure to clearly state the decision variables of the model and give their definition.

2. **Linear Optimization**

Let $P \subseteq \mathbb{R}^n$ be a polyhedron in standard form. Suppose that from any vertex of P we are only allowed to move to an adjacent vertex. Define the distance $d(x,y)$ between two vertices x and y of P as the *minimum* number of such moves required to reach y starting from x. The diameter $\delta(P)$ of P is then defined as the *maximum* of $d(x,y)$ over all pairs (x,y) of vertices of P.

(i) Explain why $\delta(P)$ is a *lower* bound on the *maximum* number of pivots the simplex algorithm might perform for minimizing a linear function over P starting from a given vertex, where the maximum is taken over all possible linear functions and starting vertices.

(ii) Let $c \in \mathbb{R}^n$ and let $\bar{x}$ be a vertex of P that is not optimal to the problem

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & x \in P. \end{aligned} \tag{1}$$

Prove that $\bar{x}$ has an adjacent vertex $\tilde{x}$ with $c^T \tilde{x} < c^T \bar{x}$.

Hint: Apply the simplex method for solving (1).

(iii) Suppose P is a $(0,1)$-polytope, i.e., a bounded polyhedron whose vertices have entries equal either to 0 or to 1. Let $\bar{x}$ be a vertex of P. Consider the problem

$$\begin{aligned} \min\ & x_n \\ \text{s.t.}\ & x \in P. \end{aligned} \tag{2}$$

Use problem (2) and the result in part (ii) to prove that, if $\bar{x}_n = 1$, then either $P \subseteq \{x \in \mathbb{R}^n : x_n = 1\}$ or there exists a vertex $\tilde{x}$ adjacent to $\bar{x}$ with $\tilde{x}_n = 0$.

(iv) Suppose P is a $(0,1)$-polytope. For $i \in \{0,1\}$ we define the polytopes in $\mathbb{R}^{n-1}$

$$P^i = \{(x_1,\dots,x_{n-1}) : (x_1,\dots,x_{n-1},i) \in P\}.$$

Let x and y be two vertices of P with $x_n = y_n = i$, $i \in \{0,1\}$. It is known that x and y are adjacent in $P \cap \{x \in \mathbb{R}^n : x_n = i\}$ —thus also in P— if and only if $(x_1,\dots,x_{n-1})$ and $(y_1,\dots,y_{n-1})$ are adjacent in P^i.

Prove that $\delta(P) \le n$.

Hint: Proceed by induction on n. For the base case, let $n = 1$. Then either $P = \{0\}$ or $P = \{1\}$. For the inductive argument, consider two vertices whose distance is $\delta(P)$ and use the result shown in (iii).

3. **Integer Optimization**

For $x \in \{0,1\}^n$, we denote by $N(x)$ the set of the n vectors in $\{0,1\}^n$ that have exactly one component different from the corresponding component of x. For example, if x is the zero vector, then $N(x)$ is the set of vectors $e_1, e_2, \ldots, e_n$ of the standard basis of $\mathbb{R}^n$.

(a) Prove that, if a linear inequality $ax \geq \beta$ is not valid for the zero vector, but is valid for $e_1, e_2, \ldots, e_n$, then it is valid for every other vector in $\{0,1\}^n$.

Hint: First show that in this case we have $a_i \geq \beta > 0$ for every $i = 1, \ldots, n$.

(b) Let $\bar{x} \in \{0,1\}^n$. Prove that, if a linear inequality $ax \geq \beta$ is not valid for $\bar{x}$, but is valid for $N(\bar{x})$, then it is valid for every other vector in $\{0,1\}^n$.

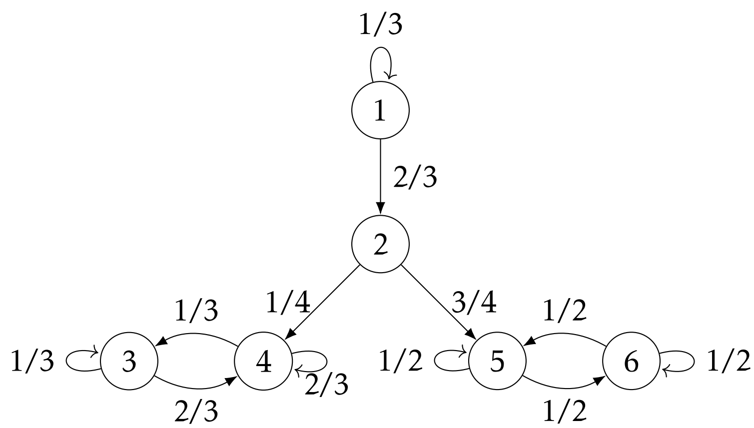
Hint: It can follow from (a) using a change of variables, or you can prove it directly.

(c) Let O be the set of vectors in $\{0,1\}^n$ that have an odd number of ones. Let $Ax \geq b$ so that $O = \{x \in \mathbb{R}^n : Ax \geq b\} \cap \{0,1\}^n$. Prove that the number of inequalities in $Ax \geq b$ is at least 2^{n-1}.

Hint: Apply (b) to each vector $\bar{x} \in \{0,1\}^n \setminus O$.

4. **Probability Modeling:**

   A discrete time Markov chain $X_n, n \geq 0$ with the state space $\{1, 2, 3, 4, 5, 6\}$ has the following transition diagram:



   Transition probabilities are given in the diagram next to corresponding transition. (E.g. $p(4,3) = 1/3$, $p(4,4) = 2/3$.)

   (a) Find the recurrent and transient states of the Markov chain. (Make sure to provide justification!)

   (b) Show that $\lim_{n\to\infty} \mathbb{P}(X_n = 3 \mid X_0 = 4)$ exists and find its value.

   (c) Suppose $X_0 = 2$. What is the long-term fraction of time that X_n spends in the set $\{1, 2, 3\}$?

5. **Queueing:** Consider a two-station queuing system *with a single server*. Customers arrive to station 1 according to a Poisson process with rate 1. If an arriving customer finds station 1 empty of other customers, they enter the system; otherwise they go away. When a customer is done at station 1, they move on to the station 2. A customer who has completed service at the station 2 leaves the system.

   The server needs an exponential amount of time with parameter 2 to complete service at either station. However, the server prioritizes station 2: if two customers are in the system, they serve the customer at station 2; if one customer is in the system, they serve the customer regardless of their station.

   (a) Draw an arrow diagram with transition rates for a Markov chain model $X_t, t \geq 0$ of this system with state space $\{0, 1, 2, 12\}$ where the state indicates the stations that are occupied. (0 means that the system is empty, and 12 means that both stations are occupied.) Don't forget to provide an explanation for your diagram!

   (b) Explain why $X_t, t \geq 0$ has a unique stationary distribution.

   (c) It costs \$10 per customer per unit time to keep a customer at either station. What is the long term average cost per unit time of the system?

   (d) Suppose that we currently have a customer at station 1 and no customers at station 2. What is the expected time until the system becomes empty?

6. A random variable X is Gamma(a,b)-distributed when it has density

$$f(x) = \frac{x^{a-1}e^{-x/b}}{\Gamma(a)b^a}.$$

Here $a > 0$ is the shape parameter and $b > 0$ is the scale parameter. We say that X is Gamma(a)-distributed when it is Gamma$(a,1)$. Note that

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}dt, \quad \Gamma(1) = 1$$

(a) Develop an algorithm for the generation of Gamma$(1,1)$ variate;

(b) If X_1 and X_2 are independent Gamma(a_1) and Gamma(a_2) random variables, show that $Y = X_1 + X_2$ is a Gamma$(a_1 + a_2)$ random variable.
Hint: Recall that the Beta function $B(t_1, t_2)$ is defined as

$$B(t_1,t_2) = \int_0^1 t^{a_1-1}(1-t)^{a_2-1},$$

and the Beta function is related to the Gamma function as

$$B(a_1,a_2) = \frac{\Gamma(a_1)\Gamma(a_2)}{\Gamma(a+1+a_2)}.$$

(c) Consider the Gamma(a) density f with parameter $(0 < a \leq 1)$. Prove that for this density, random variates can be generated by rejection from the Weibull(a) which is given as follows:

$$g(x) = ax^{a-1}e^{-x^a}, \quad (x \geq 0).$$

Hint: Note that

$$\frac{f(x)}{g(x)} = \frac{\frac{x^{a-1}e^{-x/b}}{\Gamma(a)b^a}}{ax^{a-1}e^{-x^a}} = \frac{e^{x^a-x}}{a\Gamma(a)} < \frac{e^{h-h^{1/a}}}{\Gamma(a+1)} \quad \text{where } h = a^{a/(1-a)}.$$